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ON THE POWER SERIES FOR $\log(1+z)$.*

BY T. H. GRONWALL.

Let z and ζ be complex variables, define e^ζ by the power series

$$(1) \quad e^\zeta = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!},$$

and write $1+z = e^\zeta$, or

$$z = \zeta + \frac{\zeta^2}{2!} + \cdots + \frac{\zeta^n}{n!} + \cdots.$$

By the inversion theorem for power series, there exists a unique power series in z without constant term, convergent for $|z|$ sufficiently small, and which substituted for ζ reduces the above relation to an identity. This series we now take as the definition of $\log(1+z)$; its coefficients are most readily calculated by observing that $d\zeta/dz = e^{-\zeta} = 1/(1+z)$, and we find

$$(2) \quad \log(1+z) = \zeta = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \quad \text{convergent for } |z| < 1.$$

The following two fundamental properties of the logarithm may be proved in various ways:

I. The analytic function $\log(1+z)$ defined by (2) has $z = -1$ as its only singular point at finite distance.

II. The analytic continuation of $\log(1+z)$ along a closed path winding once in the positive sense about $z = -1$ increases that function by $2\pi i$.

It is the purpose of the present note to establish these properties by actually performing the analytic continuation of the power series (2), and the proofs employed are sufficiently simple to serve as illustrations of the process of analytic continuation in an introductory course on the theory of functions of a complex variable.

It is well known that when the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$, convergent for $|z-a| < r$, is continued analytically to the point a' , where $|a'-a| < r$, the result is

$$(3) \quad \sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} c_n'(z-a')^n,$$

where

$$(4) \quad c_n' = \sum_{\nu=0}^{\infty} \frac{(n+\nu)!}{n! \nu!} c_{n+\nu} (a' - a)^\nu,$$

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and the identity (3) is valid at least for $|z - a'| < r - |a' - a|$, the latter expression being also a lower bound for the radius of convergence of the series to the right in (3). For $a = 0$, $a' = a_1$, where $|a_1| < 1$, and $c_n = (-1)^{n-1}/n$, formula (4) gives

$$\begin{aligned} c_n' &= \sum_{\nu=0}^{\infty} \frac{(n+\nu)!}{n! \nu!} \frac{(-1)^{n+\nu-1} a_1^\nu}{n+\nu} = \frac{(-1)^{n-1}}{n} \sum_{\nu=0}^{\infty} \frac{(n+\nu-1)!}{(n-1)! \nu!} (-a_1)^\nu \\ &= \frac{(-1)^{n-1}}{n} \cdot \frac{1}{(1+a_1)^n}, \quad (n \geq 1), \end{aligned}$$

the last result being obtained by making $z = -a_1$ in the expansion

$$\frac{1}{(1-z)^n} = \sum_{\nu=0}^{\infty} \frac{(n+\nu-1)!}{(n-1)! \nu!} z^\nu, \quad (n \geq 1 \text{ and } |z| < 1),$$

which is immediately verified by multiplying $n-1$ times by itself the identity

$$\frac{1}{1-z} = \sum_{\nu=0}^{\infty} z^\nu,$$

or differentiating it $n-1$ times. Thus the analytic continuation of (2) to the point a_1 , where $|a_1| < 1$, gives the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} a_1^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(1+a_1)^n} (z-a_1)^n$$

and its radius of convergence is seen at once, for instance by the ratio test, to be $|1+a_1|$.

The analytic continuation to the point a_2 , where $|a_2 - a_1| < |1+a_1|$, has as coefficient of $(z-a_2)^n$ the expression

$$\begin{aligned} \sum_{\nu=0}^{\infty} \frac{(n+\nu)!}{n! \nu!} \frac{(-1)^{n+\nu-1}}{n+\nu} \frac{(a_2-a_1)^\nu}{(1+a_1)^{n+\nu}} &= \frac{(-1)^{n-1}}{n(1+a_1)^n} \cdot \frac{1}{\left(1 + \frac{a_2-a_1}{1+a_1}\right)^n} \\ &= \frac{(-1)^{n-1}}{n} \cdot \frac{1}{(1+a_2)^n} \quad \text{for } n \geq 1, \end{aligned}$$

so that we next obtain the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} a_1^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{a_2-a_1}{1+a_1}\right)^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(1+a_2)^n} (z-a_2)^n,$$

with a radius of convergence equal to $|1+a_2|$.

Proceeding thus by successive analytic continuation to the points $a_1, a_2, \dots, a_m$, where

$$(5) \quad |a_1| < 1, \quad |a_\nu - a_{\nu-1}| < |1+a_{\nu-1}|, \quad \nu = 2, 3, \dots, m,$$

we obtain the power series

$$(6) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[a_1^n + \left(\frac{a_2 - a_1}{1 + a_1} \right)^n + \left(\frac{a_3 - a_2}{1 + a_2} \right)^n + \dots \right. \\ \left. + \left(\frac{a_m - a_{m-1}}{1 + a_{m-1}} \right)^n \right] + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(1 + a_m)^n} (z - a_m)^n,$$

with a radius of convergence equal to $|1 + a_m|$. Thus the circle of convergence of each of the analytic continuations (6) passes through the point $z = -1$. Now let z_0 be any point at finite distance distinct from -1 ; then we may join the origin to z_0 by a path not passing through -1 (for instance, the straight line from 0 to z_0 , unless z_0 lies on the negative real axis, in which case we may use the straight line from 0 to $\frac{1}{2}i$ and the straight line from $\frac{1}{2}i$ to z_0). Let l be the length of this path, $d > 0$ the minimum distance of any of its points from $z = -1$, and m an integer greater than l/d ; then dividing the length of the path into m equal parts, denoting the division points by $a_1, a_2, \dots, a_{m-1}$ and making $a_m = z_0$, we evidently have a sequence of points satisfying (5), so that the power series (2) may be continued analytically to the point $z_0 (\neq -1)$. The only singular point at finite distance is therefore $z = -1$.

To establish property II, we observe that the analytic continuations of (2) along two paths, both of which begin and end at the origin, and wind once in the positive sense about $z = -1$, yield the same result. We may therefore use the path $a_0 a_1 a_2 \dots a_m$, where

$$a_\nu = -1 + e^{2\nu\pi i/m}, \quad \nu = 0, 1, \dots, m,$$

so that $a_0 = a_m = 0$. Here π is of course defined as the smallest positive root of $e^{2\pi i} = 1$. We find that (5) is satisfied when

$$\left| \frac{a_\nu - a_{\nu-1}}{1 + a_{\nu-1}} \right| = |e^{2\pi i/m} - 1| < 1,$$

which is certainly the case for m sufficiently large, since $e^{2\pi i/m} - 1 \rightarrow 0$ as $m \rightarrow \infty$ by (1). From (6) and (2) it follows, on account of $a_0 = a_m = 0$, that the analytic continuation increases (2) by the constant

$$K = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\nu=1}^m \left(\frac{a_\nu - a_{\nu-1}}{1 + a_{\nu-1}} \right)^n,$$

or substituting the values of a_ν ,

$$K = m \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^{2\pi i/m} - 1)^n.$$

But by the remark made above regarding equivalent paths, the value of

K is independent of m , and may therefore be evaluated by letting m tend to infinity. From (1) we find

$$m(e^{2\pi i/m} - 1) \rightarrow 2\pi i \quad \text{as} \quad m \rightarrow \infty,$$

so that

$$|e^{2\pi i/m} - 1| < \frac{A}{m},$$

where A is a constant independent of m . Hence for $m > A$,

$$\left| m \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} (e^{2\pi i/m} - 1)^n \right| < m \sum_{n=2}^{\infty} \left(\frac{A}{m} \right)^n = \frac{A^2}{m - A} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty,$$

and substituting these results in the expression for K , we finally obtain

$$K = 2\pi i.$$
